Announcements

1) New Webwork up,
due next Thursday
10/20

2) Change in Amanda's
mentoring hours
No more Friday,
Th 12-2
CB 2090

Tangent Lines

Recall that for a function f, the derivative f' evaluated at $x = a$ gives the slope of the tangent line to the graph of f at $x = a$.

Formula for Tangent Line

$$y - f(a) = m(x - a)$$

$$= f'(a)(x - a)$$

Example 1. Let $f(x) = 7x^3 - 9x^2 + 115x + 2$

Find the equation of the tangent line at $x = 3$.

First, find $f'(x)$.

$f'(x) = 21x^2 - 18x + 115$.

Slope is $f'(3) = 189 - 54 + 115$
$= 250$

$f(3) = 189 - 81 + 345 + 2$
$= 455$

Equation:

$$\boxed{y - 455 = 250(x - 3)}$$

Example 2: Let $f(x) = \sec(\sqrt{x})$.

Find the equation of the tangent line at $x = \pi^2$.

$$f'(x) = \sec(\sqrt{x})\tan(\sqrt{x})\frac{1}{2}x^{-1/2}$$

(chain + power rules)

$$f'(\pi^2) = \sec(\pi)\tan(\pi)\frac{1}{2}\cdot\frac{1}{\pi}$$

$$= 1 \cdot 0 \cdot \frac{1}{2\pi} = 0$$

$$f(\pi) = \sec(\pi) = -1$$

So the equation is

$$\boxed{y + 1 = 0}$$

Higher Derivatives

Suppose f is differentiable at $x=a$. Then it may (or may not) be the case that

$$(f')'(a) = \lim_{x \to a} \frac{f'(x) - f'(a)}{x-a}$$

exists.

If the limit exists, write simply $f''(a)$ for the limit.

We call the function

$$\boxed{f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}}$$

the second derivative of f

(defined wherever the limit exists).

Example 3:

Observe that

if $f(x) = x^{4/3}$, then

$$f'(x) = \frac{4}{3}x^{1/3} \text{ for}$$

all values of x

If $x \neq 0$, $f''(x) = \frac{4}{9}x^{-2/3}$

$$= \frac{4}{9x^{2/3}}$$

However, if $x=0$,

$$f''(0) = \lim_{h \to 0} \frac{f'(0+h) - f'(0)}{h}$$

$$= \lim_{h \to 0} \frac{f'(h) - f'(0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{4}{3} h^{1/3}}{h}$$

$$= \lim_{h \to 0} \frac{4}{3h^{2/3}} = \frac{4}{0}$$

does not exist as a real number (∞).

The third derivative of f is given by

$$\boxed{f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h}}$$

provided the limit exists.

$$f^{(n)}(x) = \lim_{h \to 0} \frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}$$

is the n^{th} derivative.

Physical Interpretation

If f represents position, then

f' is <u>velocity</u>,

f'' is <u>acceleration</u>,

f''' is <u>jerk</u>.

Example 4: Suppose the (unrealistic) position of a neutrino is given by $x(t) = \dfrac{t^2 \csc(\pi \cos(t))}{\tan(t)}$ for $0 < t < \frac{\pi}{2}$.

Find the acceleration at $t = \frac{\pi}{4}$.

$$x(t) = \frac{t^2 \csc(\pi\cos(t))}{\tan(t)}$$

$$= t^2 \csc(\pi\cos(t))\cot(t)$$

(avoid quotient rule)

$$x'(t) = 2t\csc(\pi\cos(t))\cot(t) + t^2 \frac{d}{dt}\left(\csc(\pi\cos(t))\cot(t)\right)$$

$$= 2t\csc(\pi\cos(t))\cot(t) + t^2\left(\csc(\pi\cos(t))\frac{d}{dt}(\cot(t)) + \cot(t)\frac{d}{dt}\csc(\pi\cos(t))\right)$$

$$= 2t\csc(\pi\cos(t))\cot(t)$$
$$+ t^2\Big(\csc(\pi\cos(t))(-\csc^2(t))$$
$$+ \cot(t)\big(-\csc(\pi\cos(t))\cot(\pi\cos(t))$$
$$\cdot \pi(-\sin(t))\big)\Big)$$

2^{nd} derivative use mathematica

or Wolfram α.

Then plug in $\frac{\pi}{4}$ I quit.

Example 5.

Position is

now given by

$$x(t) = \cos(\pi t^3)$$

for $0 \le t \le 1$

Find acceleration at $\frac{1}{\sqrt[3]{2}} = t$.

$$x'(t) = -\sin(\pi t^3)\,\frac{d}{dt}(\pi t^3)$$

(chain rule)

$$= -\sin(\pi t^3)\cdot \pi \cdot 3t^2$$

$$x''(t) = -\sin(\pi t^3)\frac{d}{dt}(3\pi t^2)$$

$$+ 3\pi t^2 \cdot \frac{d}{dt}\left(-\sin(\pi t^3)\right)$$

(product rule)

$$= -\sin(\pi t^3)\cdot 6\pi t$$

$$+ 3\pi t^2\left(-\cos(\pi t^3)\,3\pi t^2\right)$$

$$= -\sin(\pi t^3)\,6\pi t - 9\pi^2 t^4 \cos(\pi t^3)$$

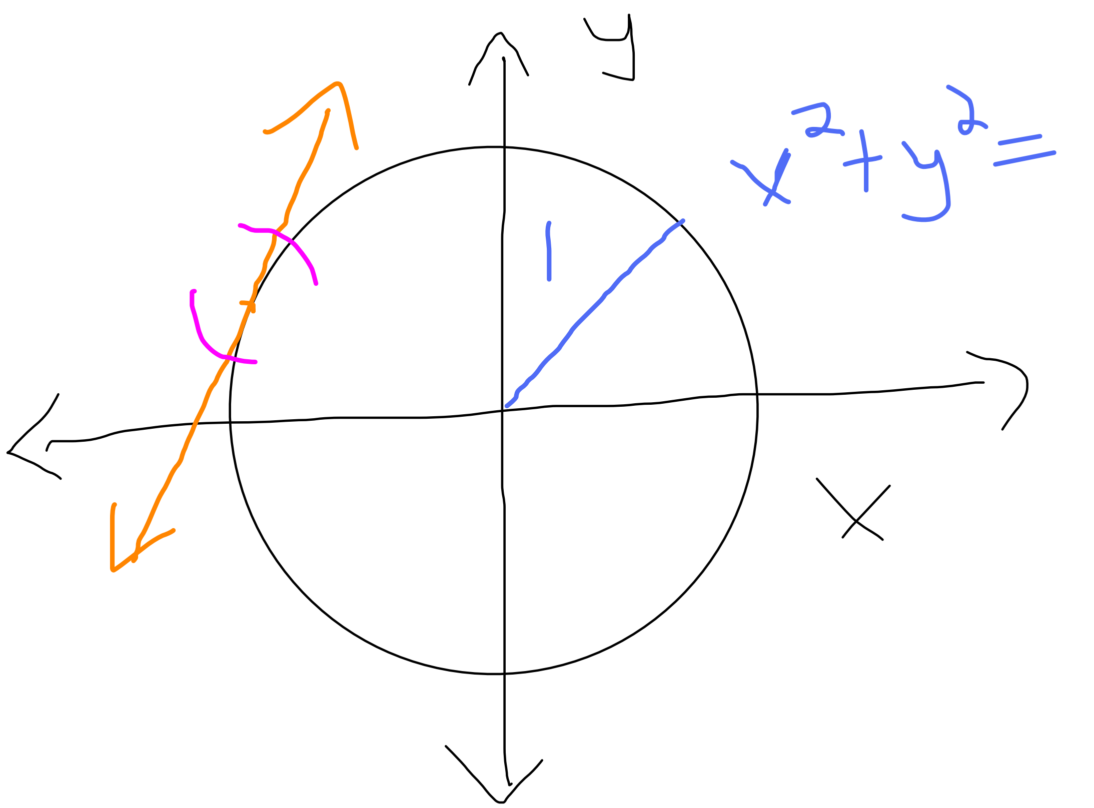
$$x''\left(\frac{1}{\sqrt[3]{2}}\right) = -\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}\frac{6\pi}{\sqrt[3]{2}}$$

$$- \frac{9\pi^2}{2^{4/3}}\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} = \boxed{-\frac{6\pi}{\sqrt[3]{2}}}$$

Implicit Differentiation

(Section 2.6)

Consider a circle $x^2+y^2=1$. Graph:



Tangent lines appear to still make sense!

<u>Idea</u>: Regard y as an "implicit" (i.e. you can't solve for y in terms of x) function of x. Provided you can do this on "small" pieces of the graph, you'll encounter no difficulties.

If $x^2+y^2=1$, then the derivatives are equal.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1) = 0$$

$$\shortparallel$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2)$$

$$\shortparallel$$

$$2x + 2y\frac{dy}{dx}$$ (chain rule)

Solve for $\frac{dy}{dx}$

$$2x + 2y\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} = -x$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Suppose we wanted the equation of the tangent line to the graph at $(1/2, -\sqrt{3}/2)$

Tangent Line at (x_0, y_0)

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$$\boxed{y - y_0 = \frac{dy}{dx}(x_0, y_0)\,(x - x_0)}$$

$$\frac{dy}{dx} = -\frac{x}{y}, \quad x_0 = \frac{1}{2}, \quad y_0 = -\frac{\sqrt{3}}{2}$$

plug in.

$$\boxed{\begin{aligned} y + \frac{\sqrt{3}}{2} &= \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}\left(x - \frac{1}{2}\right) \\ &= \frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right) \end{aligned}}$$